ON THE GENERALIZED PASCAL MATRIX-REVISITED

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ABSTRACT: In this paper, we give a proof of a theorem about the generalized Pascal matrix by mathematical induction.

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1. INTRODUCTION

In a recent paper [1], the authors discussed the sum of the Pascal matrix and the identity matrix $P_n + I_n$, where $P_n$ is the $n \times n$ Pascal matrix and $I_n$ is the $n \times n$ identity matrix. Furthermore, they obtained an explicit inverse of the Pascal matrix plus one in Theorem 1.2[1].

In [1], the authors defined that for any complex number $a$, let $P(a)$ denote the generalized Pascal matrix, which is written in the form of an infinite, lower triangular matrix as:

$$P(a) = \begin{bmatrix}
1 \\
a \\
a^2 & 2a & 1 \\
a^3 & 3a^2 & 3a & 1 \\
\vdots & \vdots & \vdots & \ddots \\
a^n & \binom{n}{1}a^{n-1} & \binom{n}{2}a^{n-2} & \cdots & \binom{n}{n-1}a & 1
\end{bmatrix}.$$

For convenient, let

$$P_{n+1}(a) = \begin{bmatrix}
1 \\
a \\
a^2 & 2a & 1 \\
a^3 & 3a^2 & 3a & 1 \\
\vdots & \vdots & \vdots & \ddots \\
a^n & \binom{n}{1}a^{n-1} & \binom{n}{2}a^{n-2} & \cdots & \binom{n}{n-1}a & 1
\end{bmatrix}.$$

In this paper, we give a proof of Corollary 2 in [1] about the generalized Pascal matrix by mathematical induction.
2. MAIN PROOF ABOUT THE INVERSE OF THE GENERALIZED PASCAL MATRIX

Now we are in a position to give the proof of Corollary 2 in [1] about the generalized Pascal matrix by mathematical induction.

**Corollary 2.** \( P_{n+1}(a)^k = P_{n+1}(ka) \).

**Proof.** We use mathematical induction on \( k \).

1. The case \( k = 1 \):

\[
P_{n+1}(a)^1 = P_{n+1}(1 \times a).
\]

2. The case \( k \Rightarrow k + 1 \) with \( k > 1 \): Assuming

\[
P_{n+1}(a)^k = P_{n+1}(ka).
\]

Then we want to show that

\[
P_{n+1}(a)^{k+1} = P_{n+1}(a)^k P_{n+1}(a) = P_{n+1}(ka) P_{n+1}(a)
\]

That is to show that

\[
P_{n+1}((k+1)a)(i, j) = P_{n+1}(ka) P_{n+1}(a)(i, j).
\]

i.e.

\[
\binom{i}{j} (i+1)(i+1)a^{i-j} = \sum_{l=1}^{i} P_{n+1}(ka)(i, l) P_{n+1}(a)(l, j) = \sum_{l=1}^{i} \binom{i}{l} (ka)^{i-l} \binom{a^{l-j}}{i-j}.
\]

\[
P_{n+1}(a)^{k+1}(i, j) = \sum_{l=1}^{i} P_{n+1}(ka)(i, l) P_{n+1}(a)(l, j) = \sum_{l=1}^{i} \binom{i}{l} (ka)^{i-l} \binom{a^{l-j}}{i-j}.
\]

Because

\[
P_{n+1}((k+1)a)(i, j) = \begin{cases} 
0, & i < j, \\
1, & i = j, \\
\binom{a^{i-j}}{i-j}(k+1)a^{i-j}, & i > j.
\end{cases}
\]
\[ P_{n+1}(a)^{k+1}(i, j) = (P_{n+1}(a)^k P_{n+1}(a))(i, j) = \sum_{l=1}^{i} P_{n+1}(ka)(i, l)P_{n+1}(a)(l, j) \]

\[
= \begin{cases} 
0, & i < j. \\
1, & i = j. \\
\sum_{l=1}^{i} (\binom{i-1}{l-1})(ka)^{i-l}(\binom{j-1}{l-1})a^{l-j}, & i > j.
\end{cases}
\]

Then

\[
P_{n+1}(a)^{k+1}(i, j) = \sum_{l=1}^{i} P_{n+1}(ka)(i, l)P_{n+1}(a)(l, j) = \sum_{l=1}^{i} P_{n+1}(ka)(i, l)P_{n+1}(a)(l, j) \]

\[
= \sum_{l=j}^{i} (\binom{i-1}{l-1})(ka)^{i-l}(\binom{j-1}{l-1})a^{l-j} = \sum_{l=j}^{i} \frac{(i-1)!}{(l-1)!(i-l)!} \frac{(l-1)!}{(j-1)!(l-j)!} (ka)^{i-l}(\binom{j-1}{l-1})a^{l-j} \]

\[
= \frac{(i-1)!}{(j-1)!(i-j)!} \sum_{l=j}^{i} \frac{(i-j)!}{(i-l)!(l-j)!} (ka)^{i-l}(\binom{j-1}{l-1})a^{l-j} \]

\[
= \frac{(i-1)!}{(j-1)!(i-j)!} (ka + a)^{i-j} = (\binom{i-1}{j-1})(k + 1)a^{i-j}.
\]

So we complete the proof.

REFERENCES
