ON OSCILLATION AND ASYMPTOTIC BEHAVIOR OF HIGHER ORDER NONLINEAR DELAY DIFFERENCE EQUATIONS

R. Arul¹ and R. Kodeeswaran²

¹,²Department of Mathematics, Kandaswami Kandar’s College, Velur - 638 182, Namakkal Dt. Tamil Nadu, India.
¹E-mail: drrarul@gmail.com, ²srkodeesh@gmail.com

ABSTRACT: In this paper we obtain criteria for the oscillation and asymptotic behavior of solutions of higher order delay difference equation of the form

\[ \Delta \left( a_n \left( \Delta^{m-1} x_n \right)^\alpha \right) = q_n f \left( x(\tau(n)) \right), \]

where \( \{a_n\}, \{q_n\} \) and \( \{\tau(n)\} \) are nonnegative real sequences, \( \alpha \) is the ratio of odd positive integers. Some examples are provided to illustrate the main results.

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1. INTRODUCTION

In this paper, we study with the following higher order delay difference equation of the form

\[ \Delta \left( a_n \left( \Delta^{m-1} x_n \right)^\alpha \right) = q_n f \left( x(\tau(n)) \right), n \in N_0 \]

where \( N_0 = \{n_0, n_0 + 1, n_0 + 2, ...\} \), \( n_0 \) is a non negative integer, \( f : R \to R \) is continuous and nondecreasing and \( \Delta \) is the forward difference operator defined by \( \Delta x_n = x_{n+1} - x_n \), subject to the following conditions:

(H1) \( \alpha \) is the ratio of odd positive integers;
(H2) \( \{a_n\} \) and \( \{q_n\} \) are nonnegative real sequences;
(H3) \( \{\tau(n)\} \) is non decreasing sequence of integers, and \( \tau(n) \leq n \) with \( \tau(n) \to \infty \) as \( n \to \infty \).
(H4) \( uf(u) > 0 \), for \( u \neq 0 \), and \( -f(uv) \geq f(uv) \geq f(u)f(v) \) for all \( uv > 0 \).

By a solution of equation (1.1), we mean a real sequence \( \{x_n\} \) which is defined for \( n \geq n_0 \) and satisfies equation (1.1) for all \( n \in N_0 \). A nontrivial solution \( \{x_n\} \) of equation (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative and nonoscillatory otherwise.

The oscillation of higher order difference equations have been investigated by many authors, see for examples [7, 8, 11] and references quoted therein. In [9, 12] the authors studied the oscillatory and asymptotic behavior of higher order functional difference equations. Following this trend, in this paper by establishing new comparison theorem for comparing higher order difference equation (1.1) with the set of first order delay difference equation, so that the oscillation of these equations yield the oscillation of equation (1.1). We will discuss both cases,

\[ \sum_{s=N_0}^{\infty} a_s^{-1/\alpha} = \infty \]
and \[ \sum_{s=N_0}^{n} a_s^{-1/\alpha} < \infty. \] (1.3)

In Section 2, we obtain some important results on the nonoscillatory properties of equation (1.1), and in Section 3, we provide some examples to illustrate the main results.

2. MAIN RESULTS

We begin with the following lemmas;

**Lemma 2.1:** If \( \{x_n\} \) is a positive solution of equation (1.1) then \( a_n(\Delta^{m-1}x_n)^\alpha \) is increasing, all derivatives \( \Delta^i(x_n) \) \( 1 \leq i \leq m-1 \) are constant sign, and \( \{x_n\} \) satisfies either

(I) \( x_n\Delta^{m-2}x_n > 0, \ x_n\Delta^{m-1}x_n < 0; \) or

(II) \( x_n\Delta x_n > 0, \ldots, x_n\Delta^{m-2}x_n < 0, \ x_n\Delta^{m-1}x_n > 0; \) or if (1.3) holds

(III) \( x_n\Delta^{m-2}x_n < 0, \ x_n\Delta^{m-1}x_n > 0. \)

**Proof:** Let \( \{x_n\} \) is a positive solution of equation (1.1), for all \( n \geq n_0. \) It follows from equation (1.1)

\[ \Delta(a_n(\Delta^{m-1}x_n)^\alpha) > 0. \]

Thus \( a_n(\Delta^{m-1}x_n) > 0 \) or \( a_n(\Delta^{m-1}x_n) < 0 \) and all lower derivatives are of fixed signs. Suppose we assume that \( a_n(\Delta^{m-1}x_n) < 0, \) then we are lead to \( a_n(\Delta^{m-2}x_n) > 0, \) because the condition \( a_n(\Delta^{m-1}x_n) < 0, \) which is contradicts the positivity of \( x_n. \) Therefore we conclude that \( \{x_n\} \) satisfies the case (I).

Now, we suppose that \( a_n(\Delta^{m-1}x_n) > 0, \) then either \( a_n(\Delta^{m-2}x_n) < 0 \) or \( a_n(\Delta^{m-2}x_n) > 0. \) From equation (1.1) we have \( a_n(\Delta^{m-1}x_n)^\alpha \) is increasing then there is a constant \( c > 0 \) such that

\[ a_n(\Delta^{m-1}x_n)^\alpha \geq c, \ n \geq N_0. \]

Summing the last inequality from \( N_0 \) to \( n - 1 \) we obtain

\[ \Delta^{m-2}x_n \geq \Delta^{m-2}x_{N_0} + c^{1/\alpha} \sum_{s=N_0}^{n-1} a_s^{-1/\alpha}. \]

If (1.2) holds, then the last inequality implies, \( a_n\Delta^{m-2}x_n > 0, \) which guarantees that all lower derivatives are positive. Hence \( \{x_n\} \) satisfies case (II). On the other hand, if (1.3) holds then we get \( a_n\Delta^{m-2}x_n < 0, \) so that \( \{x_n\} \) may satisfies case (III). This complete the proof.

**Definition 2.1:** A nonoscillatory solution \( \{x_n\} \) of equation (1.1) is said to be strongly increasing if \( \{x_n\} \) is positive and it satisfies case (II) of Lemma 2.1, or \( \{x_n\} \) is negative and \( -\{x_n\} \) satisfies case (II) of Lemma 2.1.

**Definition 2.2:** Assume that (1.2) holds. We say that equation (1.1) has property (B) if every its nonoscillatory solution \( \{x_n\}, \)
(i) for $m$ odd, is strongly increasing,

(ii) for $m$ even, is strongly increasing or satisfies $\lim_{n \to \infty} \{x_n\} = 0$.

**Remark 2.1:** It is easy to verify that, if $\{x_n\}$ is strongly increasing then $\{x_n\}$ holds the following,

$$x_n \geq c \sum_{s=N_0}^{n-1} a_s^{-1/\alpha} (n-s)^{-\alpha}, \quad c > 0. \quad (2.1)$$

**Lemma 2.2:** Let $y_n$ be defined for $n \geq n_0 \in N_0$ and $y_n > 0$ with $\Delta^m y_n$ of constant sign for $n \geq n_0, m \in N_0$ and not identically zero. Then there exists an integer $k, 0 \leq k \leq m$ with $m + k$ even for $\Delta^m y_n \geq 0$ or $m + k$ odd for $\Delta^m y_n \leq 0$ such that

(i) $k \leq m - 1$ implies $(-1)^{k+i} \Delta^i y_n > 0$ for all $n \geq n_0, \, k \leq i \leq m - 1$,

(ii) $k \geq 1$ implies $\Delta^i y_n > 0$ for all $n \geq n_0, \, 1 \leq i \leq k - 1$.

**Lemma 2.3:** Let $y_n$ be defined for $n \geq n_0$, and $y_n > 0$ with $\Delta^m y_n \leq 0$ for $n \geq n_0$ and not identically zero. Then there exists a large $n_1 \geq n_0$ such that

$$y_n \geq \frac{1}{(m-1)!} (n-n_1)^{m-1} \Delta^{m-1} y(2^{m-k-1}(n)), \quad n \geq n_1,$$

where $m$ is defined as in Lemma 2.2. Further if $y_n$ is increasing, then

$$y_n \geq \frac{1}{(m-1)!} \left( \frac{n}{2^{m-1}} \right)^{m-1} \Delta^{m-1} y_n, \quad n \geq 2^{m-1} n_1. \quad (2.2)$$

The proof of Lemmas 2.2 and 2.3 can be found in [3].

**Theorem 2.1:** Assume that $\sigma(n)$ is nondecreasing sequence of integer such that,

$$\sigma(n) > n \quad \text{and} \quad \tau(\sigma(n)) < n. \quad (2.3)$$

Further assume that $\{x_n\}$ is a positive solution of equation (1.1) such that $\lim_{n \to \infty} x_n \neq 0, \lim_{n \to \infty} x_n \neq 0$. If for some $\delta \in (0, 1)$ the first order delay difference equation

$$\Delta y_n + a_n^{-1/\alpha} \sum_{s=n_0}^{\sigma(n)-1} q_s f \left( \frac{\delta(\tau(s))^{m-2}}{(m-2)!} \right)^{1/\alpha} f^{1/\alpha} y(\tau(\sigma(n))) = 0. \quad (2.4)$$

is oscillatory, then $\{x_n\}$ does not satisfy case (I) of Lemma 2.1.

**Proof:** Let $\{x_n\}$ be a positive solution of equation (1.1) satisfies case (I) of Lemma 2.1. It follows from Lemma 2.3 that for every $\delta \in (0, 1)$

$$x_n \geq \frac{\delta_{n_0}^{m-2}}{(m-2)!} \Delta^{m-2} x_n. \quad (2.5)$$
Using (2.5) in equation (1.1) we get

\[ \Delta(a_n \Delta^{m-1} x_n)^\alpha \geq q_n f \left( \frac{\delta(s)^{m-2} x^{(\tau(s))}}{(m-2)!} \right)^{m-2} x^{(\tau(s))} \]

\[ \geq q_n f \left( \frac{\delta(s)^{m-2}}{(m-2)!} \right) f(\Delta^{m-2} x^{(\tau(s)))}. \] (2.6)

Summing (2.6) from \( n \) to \( \sigma(n) - 1 \), we obtain

\[ -a_n \Delta^{m-1} x_n^\alpha \geq \sum_{s=n}^{\sigma(n)-1} q_s f \left( \frac{\delta(s)^{m-2}}{(m-2)!} \right) f(\Delta^{m-2} x^{(\tau(s)))}. \]

\[ \geq f(\Delta^{m-2} x^{(\tau(\sigma(n)))}) \sum_{s=n}^{\sigma(n)-1} q_s f \left( \frac{\delta(s)^{m-2}}{(m-2)!} \right), \]

where we have used the monotonicity of \( f(\Delta^{m-1} x^{(\tau(s)))}. \) Consequently, \( y_n = \Delta^{m-2} x_n \) is a positive solution of delay difference inequality

\[ \Delta y_n + a_n^{1/\alpha} \left[ \sum_{s=n}^{\sigma(n)-1} q_s f \left( \frac{\delta(s)^{m-2}}{(m-2)!} \right) \right]^{1/\alpha} y^{(\tau(\sigma(n)))} \leq 0. \] (2.7)

It follows from Theorem 3 in [10], the corresponding difference equation (2.4) has a positive solution, which is a contradiction. Hence we conclude that \( \{x_n\} \) can not satisfy case(I) of Lemma 2.1. This complete the proof.

**Theorem 2.2:** Let (1.2) holds. If for some \( \delta \in (0, 1) \), then first order difference equation (2.4) is oscillatory then equation (1.1) has property (B).

**Proof:** Let \( \{x_n\} \) be a nonoscillatory solution of equation (1.1). We may assume that \( x_n > 0 \). It follows from Lemma 2.1 that for \( x_n \) satisfies either case (I) or case (II). We first assume that \( m \) is odd. We shall show that \( \{x_n\} \) is strongly increasing. That is \( \{x_n\} \) satisfies case (II) of Lemma 2.1.

Assume the contradiction, let \( \{x_n\} \) satisfies case (I) of Lemma 2.1. Then it follows from case (I) of Lemma 2.1 that \( x_n > 0 \), therefore \( \lim_{n \to \infty} x_n \neq 0 \). But Theorem 2.1 shows that \( \{x_n\} \) does not satisfies case (I) of Lemma 2.1. We conclude that \( (x_n) \) satisfies case (II) of Lemma 2.1. Now we assume \( m \) is even. We shall show that \( \{x_n\} \) is strongly increasing or \( \lim_{n \to \infty} x_n = 0 \).

If we assume that \( \lim_{n \to \infty} x_n \neq 0 \) then by Theorem 2.1, the oscillation of equation (2.4) yield that \( \{x_n\} \) is strongly increasing. This complete the proof.

As a consequence of Theorem 2.2, we derive the following corollaries.

**Corollary 2.1:** Let (1.2) and (2.3) hold. If

\[ \frac{f(u^{1/\alpha})}{u} \geq 1, \quad 0 < |u| \leq 1 \]
and for some $\delta \in (0,1)$

$$\liminf_{n \to \infty} \sum_{u=\tau(n)}^{n-1} a_u^{-1/\alpha} \left[ \sum_{s=u}^{\sigma(u)-1} q_s \left( \frac{\delta(\tau(s))^{m-2}}{(m-2)!} \right)^{1/\alpha} \right] \geq \left( \frac{\alpha}{\alpha + 1} \right)^{\alpha+1},$$

(2.9)

then equation (1.1) has property (B).

**Corollary 2.2:** Let (1.2) and (2.3) hold, $\beta$ is the ratio of odd positive integers and $\beta < \alpha$. If

$$\limsup_{n \to \infty} \sum_{u=\tau(n)(n))}^{n-1} a_u^{-1/\alpha} \left[ \sum_{s=u}^{\sigma(u)-1} q_s ((\tau(s))^{m-2})^{\beta} \right]^{1/\alpha} > 0$$

then the difference equation

$$\Delta(a_n (\Delta^{-1} x_n)^\alpha) + q_n x^\beta(\tau(n)) = 0$$

(2.11)

has property (B).

**Theorem 2.3:** Let $\{x_n\}$ be a positive solution of equation (1.1) such that $\lim_{n \to \infty} x_n \neq 0$. If for some $\delta \in (0,1)$ the first order delay difference equation

$$\Delta y_n + a_n^{-1/\alpha} \left[ \sum_{s=N_1}^{n-1} q_s f \left( \frac{\delta(\tau(s))^{m-2}}{(m-2)!} \right)^{1/\alpha} \right] f^{1/\alpha} y(\tau(n)) = 0,$$

(2.12)

is oscillatory then $\{x_n\}$ does not satisfies case (III) of Lemma 2.1.

**Proof:** Assume the contrary. Suppose that $\{x_n\}$ satisfies case (III) of Lemma 2.1.

By Lemma 2.2, we see that for every $\delta \in (0,1)$

$$x_n \geq -\frac{\delta(n)^{m-2}}{(m-2)!} \Delta^{m-2} x_n.$$  

(2.13)

Using (2.13) in equation (1.1) we obtain,

$$\Delta(a_n (\Delta^{-1} x_n)^\alpha) \geq q_n f \left( \frac{\delta(\tau(n))^{m-2}}{(m-2)!} \right) f(-\Delta^{m-2} x(\tau(n))).$$

(2.14)

Summing inequality (2.14) from $N_1$ to $n-1$ we have

$$a_n (\Delta^{-1} x_n)^\alpha \geq \sum_{s=N_1}^{n-1} q_s f \left( \frac{\delta(\tau(s))^{m-2}}{(m-2)!} \right) f(-\Delta^{m-2} x(\tau(s)))$$

$$\geq f(-\Delta^{m-2} x(n)) \sum_{s=N_1}^{n-1} q_s f \left( \frac{\delta(\tau(s))^{m-2}}{(m-2)!} \right),$$

(2.15)
where we have used the monotonicity of \( f(-\Delta^{m+2} x(\tau(n))) \) and consequently \( y_n = -\Delta^{m+2} x_n \) is a positive solution of the delay difference inequality

\[
\Delta y_n + a_n^{-1/\alpha} \left[ \sum_{s=N_1}^{n-1} q_s f \left( \frac{\delta(s)^{\nu-2}}{(m-2)!} \right) \right]^{\gamma/\alpha} y(\tau(n)). \tag{2.16}
\]

It follows from Theorem 3 in [10] that the corresponding difference equation (2.12) has a positive solution, which is a contraction. Hence we conclude that \( \{x_n\} \) can not satisfies case (III) of Lemma 2.1. The proof is now complete.

**Theorem 2.4:** Let (1.3) holds. If for some constant \( \delta \in (0, 1) \) both first order delay difference equations (2.4) and (2.12) are oscillatory, then every nonoscillatory solution \( \{x_n\} \) of equation (1.1) is strongly increasing or satisfies \( \lim_{n \to \infty} x_n = 0. \)

**Proof:** Assume that \( \{x_n\} \) is a nonoscillatory solution of equation (1.1) and assume with out loss of generality that \( x_n > 0 \) for all \( n \geq n_0. \) It follows from Lemma 2.1 that \( \{x_n\} \) satisfies either case (I) or case (II) or case (III) of Lemma 2.1. Suppose assume that \( \lim_{n \to \infty} x_n \neq 0, \) then by Theorems 2.1 and 2.3 oscillation of equation (2.4) and (2.11) doesnot satisfies case (I) and case (III) of Lemma 2.1 respectively. Therefore we conclude that \( \{x_n\} \) has to satisfies case (II) of Lemma 2.1. This complete the proof.

**Corollary 2.3:** Let (1.3), (2.3) and (2.8) are hold. If for some \( \delta \in (0, 1), \) and (2.9) holds and

\[
\lim_{n \to \infty} \inf \sum_{u=\tau(n)}^{n-1} a_u^{-1/\alpha} \left[ \sum_{s=N_1}^{n-1} q_s f \left( \frac{\delta(s)^{\nu-2}}{(m-2)!} \right) \right]^{\gamma/\alpha} \geq \left( \frac{\alpha}{\alpha+1} \right)^{\alpha+1} \tag{2.17}
\]

then every nonoscillatory solution of equation (1.1) is strongly increasing or \( \lim_{n \to \infty} x_n = 0. \)

**Proof:** It follows from Theorem 2.4 and equation (2.17), we conclude that equation (1.1) is strongly increasing or satisfies \( \lim_{n \to \infty} x_n = 0. \)

### 3. EXAMPLES

In this section, we present three examples to illustrate the main results.

**Example 3.1:** Consider the higher order nonlinear delay difference equation

\[
\Delta(2^n(\Delta^3 x_n)^3) = \frac{3}{2^{n+4}} x_{n-3}, \quad n \geq 4, \tag{3.1}
\]

where \( a_n = 2^n, \alpha = 3, \tau(n) = n - 3 \) and \( q_n = \frac{3}{2^{n+4}}. \)

We set \( \sigma(n) = n + 2, \) it is easy to verify that condition (2.9) holds. From Corollary 2.1 we see that the equation (3.1) having property (B). In fact \( \{x_n\} = \left\{ \frac{1}{2^n} \right\} \) is one such solution of equation (3.1).

**Example 3.2:** Consider the higher order delay difference equation
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\[ \Delta \left( \frac{1}{n} \Delta^2 x_n \right) = \frac{8}{n(n+1)(n-2)^2} x_{n-2}, \quad n \geq 3, \quad (3.2) \]

where \( a_n = \frac{1}{n}, \alpha = 3, \tau(n) = n - 2 \) and \( q_n = \frac{8}{n(n+1)(n-2)^2} \).

We set \( \sigma(n) = n + 1 \), it is easy to verify that condition (2.17) holds. From Corollary 2.3 we see that the equation (3.2) having property (B). In fact \( \{x_n\} = \{\frac{1}{n^2}\} \) is one such solution of equation (3.2).

Example 3.3: Consider the higher order delay difference equation

\[ \Delta((\Delta^3 x_n)^2) = \frac{7}{2^{2n+15}} x_{n-3}, \quad n \geq 4, \quad (3.3) \]

where \( a_n = 1, \alpha = 3, \beta = 1, \tau(n) = n - 3 \) and \( q_n = \frac{7}{2^{2n+15}} \).

We set \( \sigma(n) = n + 1 \), it is easy to verify that condition (2.10) holds. From Corollary 2.2 we see that the equation (3.3) having property (B). In fact \( \{x_n\} = \{-\frac{1}{2^n}\} \) is one such solution of equation (3.3).

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