ON HESSENBERG AND PENTADIAGONAL DETERMINANTS-REVISED

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ABSTRACT: In this paper, we establish a different proof of the determinant related to one kind of generalized Fibonacci sequence.

Keywords: Fibonacci numbers; Hessenberg determinants.

1. INTRODUCTION

In a recent paper [1], the authors defined two generalized Fibonacci and Lucas sequences as following:

For any integer number $s > 0$ and $t \neq 0$ with $s^2 + 4t > 0$, the $n$th $(s, t)$-Fibonacci sequence, say $\{F_n(s, t)\}_{n \in \mathbb{N}}$ is defined recurrently by

$$F_{n+1}(s, t) = sF_n(s, t) + tF_{n-1}(s, t) \quad \text{for} \quad n \geq 1, \quad \text{with} \quad F_0(s, t) = 0, \quad F_1(s, t) = 1.$$ 

For any integer number $s > 0$ and $t \neq 0$ with $s^2 + 4t > 0$, the $n$th $(s, t)$-Lucas sequence, say $\{L_n(s, t)\}_{n \in \mathbb{N}}$ is defined recurrently by

$$L_{n+1}(s, t) = sL_n(s, t) + tL_{n-1}(s, t) \quad \text{for} \quad n \geq 1, \quad \text{with} \quad L_0(s, t) = 0, \quad L_1(s, t) = 1.$$ 

In this paper, we give a different result of Theorem 3 about the determinant of matrices[1] related to $(s, t)$-Fibonacci sequences.

2. THE MAIN PROOF OF A DIFFERENT RESULT OF THE DETERMINANT RELATED TO ONE KIND OF GENERALIZED FIBONACCI SEQUENCES

Now we are in a position to establish a different result of Theorem 3 about the determinant of matrices[1] related to $(s, t)$-Fibonacci sequences.

Theorem 1. For any integers numbers $s > 0$ and $t \neq 0$ with $s^2 + 4t > 0$, define the $(n+1) \times (n+1)$ matrix $H_{n+1}^{*}$ as

$$H_{n+1}^{*} = \frac{1}{n!} \det \begin{bmatrix} F_{2n} & 2tF_{2n-1} & (2t)^2F_{2n-2} & \cdots & (2t)^{n-1}F_{n+1} & (2t)^nF_n \\ n & -s & (2t)^2 & \cdots & (2t)^{n-1} & (2t)^n \\ n-1 & -2s & n-2 & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\ 1 & -ns & \cdots & \cdots & \cdots & -ns \end{bmatrix}, \quad n \geq 0.$$ 

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Then, the determinant $H_{n+1}^*$ is given by

$$H_{n+1}^* = (-1)^n \left[ \sum_{i=0}^{n} (C_i^n) s^{n-i} (2t)^i F_{2n-i} \right].$$

**Proof:** By expanding the matrix along the first row, we get

$$H_{n+1}^* = \frac{1}{n!} [F_{2n} \det ]
\begin{bmatrix}
-s \\
n-1 & -2s & 0 \\
n-2 & -3s \\
0 & \ddots & \ddots \\
1 & -ns_{rev}
\end{bmatrix}
$$

$$-(2t)F_{2n-1} \det
\begin{bmatrix}
n \\
0 & -2s & 0 \\
n-2 & -3s \\
n-3 & -4s \\
0 & \ddots & \ddots \\
1 & -ns_{rev}
\end{bmatrix}
$$

$$+(2t)^2 F_{2n-2} \det
\begin{bmatrix}
n & -s \\
0 & n-1 & 0 \\
0 & -3s \\
n-3 & -4s \\
0 & \ddots & \ddots \\
1 & -ns_{rev}
\end{bmatrix}
$$

$$-(2t)^3 F_{2n-3} \det
\begin{bmatrix}
n & -s \\
n-1 & -2s & 0 \\
n-2 & \ddots & \ddots \\
n-4 & -5s \\
0 & \ddots & \ddots \\
1 & -ns_{rev}
\end{bmatrix}
$$
\[
\begin{bmatrix}
1 & 2 & 0 \\
& 1 & 2 & 0 \\
& & \ddots & \ddots \\
& & & 1 & 2 & 0 \\
& & & & \ddots & \ddots \\
& & & & & 1 & 2 & 0 \\
\end{bmatrix}
\]

\[= \frac{1}{n!} \left[ F_{2n} \times (-1)^n n! \times s^n - (2t) F_{2n-1} \times (-1)^{n-1} n \times n s^{n-1} + (2t)^2 F_{2n-2} \times (-1)^{n-2} \frac{n(n-1)}{2} s^{n-2} \times n! \right.
\]

\[-(2t)^3 F_{2n-3} \times (-1)^{n-3} \frac{n(n-1)(n-2)}{3!} n \times s^{n-3} + \cdots + (2t)^k F_{2n-k} \times (-1)^{n-k} \frac{n(n-1) \cdots ((n-k)+1)}{k!} s^{n-k} \times n! \]

\[+ \cdots + (-1)^n (2t)^n F_n \times n! \]

\[= (-1)^n \left[ \sum_{l=0}^{n} (C_l^n) s^{n-l} (2t)^l F_{2n-2l} \right]. \]

References
