EXISTENCE OF NONOSCILLATORY SOLUTIONS OF FIRST ORDER NONLINEAR NEUTRAL DELAY DIFFERENCE EQUATIONS

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ABSTRACT: In this paper the authors consider the first order nonlinear neutral delay difference equation
\[ \Delta (x(n) + p(n) x(n - \tau)) + q_1(n) x(n - \sigma_1) + q_2(n) x(n - \sigma_2) = e(n) \]
where \( \tau, \sigma_1, \sigma_2 \) are positive integers, \( p(n), q_1(n), q_2(n) \) and \( e(n) \) are real sequences. By using Krasnoselskii’s fixed point theorem, we obtain sufficient conditions for the existence of nonoscillatory solutions. The Examples are illustrated with MATLAB Programming.


KEYWORDS AND PHRASES: Difference equations, Nonoscillation, Krasnoselskii’s Fixed Point Theorem.

1. INTRODUCTION

In this paper, we consider the first order nonlinear neutral delay difference equation
\[ \Delta (x(n) + p(n) x(n - \tau)) + q_1(n) x(n - \sigma_1) + q_2(n) x(n - \sigma_2) = e(n) \]  \hspace{1cm} (1)
where \( \Delta \) is the forward difference operator defined by \( \Delta x(n) = x(n) - x(n - 1) \), \( \tau, \sigma_1, \sigma_2 \) are positive integers, \( p(n), q_1(n), q_2(n) \) and \( e(n) \) are real sequences defined for all \( n \in \mathbb{N}(n_0) = \{n_0, n_0 + 1, n_0 + 2, \ldots \} \), \( n_0 \) a positive integer. Let \( \rho = \max \{\tau, \sigma_1, \sigma_2\} \). By a solution of equation (1), we mean a real sequence \( x(n) \) defined for all \( n \geq \mathbb{N}(n_0 - \rho) \) and satisfies equation (1).

Oscillation and nonoscillation properties of solutions difference equations have developed very rapidly in recent years refer [1-4]. There has been growing interest in the study of discrete time models described by difference equations which arise quite naturally in population dynamics, epidemic models and electronic circuit analysis [5], [6], [7]).

A solution of the difference equation (1) is called eventually positive if there exists a positive integer \( n_0 \) such that \( x(n) > 0 \) for \( n \in \mathbb{N}(n_0) \). If there exists a positive integer \( n_0 \) such that \( x(n) < 0 \) for \( n \in \mathbb{N}(n_0) \), then (1) is called eventually negative.

The solution of the difference equation (1) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is called nonoscillatory.

2. PRELIMINARY RESULTS

In this paper, using Krasnoselskii’s fixed point theorem, we obtain some sufficient conditions for the existence of a bounded nonoscillatory solution of equation (1).
The space $\ell^\infty$ is the set of real sequences defined on the set of positive integers where any individual sequence is bounded with respect to the usual supremum norm. It is well known that under the supremum norm $\ell^\infty$ is a Banach space.

**Definition 2.1:** A subset $\Omega$ of a Banach Space $X$ is relatively compact if every sequence in converges to an element of $X$.

**Definition 2.2:** A set $S$ of sequences in $\ell^\infty$ is uniformly Cauchy (or equi-Cauchy) if for every $\epsilon > 0$, there exists an integer $N$ such that $|x_i - x_j| < \epsilon$ whenever $i, j > N$ for any $x = \{x_i\}$ in $S$.

**Lemma 2.3:** (Discrete Arzela-Ascoli’s Theorem) A bounded, uniformly Cauchy subset $\Omega$ of $\ell^\infty$ is relatively compact.

**Theorem 2.4:** (Krasnoselskii’s Fixed Point Theorem). Let $X$ be a Banach space. Let be a bounded convex subset of $X$ and let $S_1, S_2$ be maps of $\ell^\infty$ into $X$ such that $S_1x + S_2y \in \Omega$ for every pair $x, y \in \Omega$. If $S_1$ is a contractive and $S_2$ is completely continuous then the equation $S_1x + S_2x = x$ has a solution in $\Omega$.

### 3. Existence of Nonoscillatory Solutions

In this section we establish sufficient condition for the existence of bounded nonoscillatory solution of equation (1).

**Theorem 3.1:** Suppose that there exist nonnegative constants $c_1$ and $c_2$ such that $c_1 + c_2 < 1$, $-c_2 \leq p(n) \leq c_1$. Further, assume that $\sum_{s = n_0}^{\infty} q_i(s) < \infty$, $i = 1, 2$. Then equation (1) has a bounded nonoscillatory solution.

**Proof:** We choose a $n_1 > n_0$ sufficiently large such that

$$\sum_{s = n_1}^{\infty} |q_1(s)| + M |q_2(s)| + |\epsilon(s)| < \frac{1-c_1-c_2}{4}, \quad (2)$$

$$\sum_{s = n_1}^{\infty} |q_i(s)| < \frac{1-c_1}{2}, \quad (3)$$

where $M = \max_{1 \leq s \leq x, y \leq 1} \{xy\}$. Let $\ell^\infty_{n_0}$ be the set of all real sequences $x = \{x_n\}_{n=n_0}^{\infty}$ with the norm $\|x\| = \sup_{n \geq n_0} |x(n)| < \infty$. Then $\ell^\infty_{n_0}$ is a Banach space. We define a closed, bounded and convex subset of $\Omega$ of $\ell^\infty_{n_0}$ as follows.

$$\Omega = \left\{ x = \{x(n)\}_{n=n_0}^{\infty} : \frac{1-c_1-c_2}{2} \leq x(n) \leq 1, n \geq n_0 \right\}.$$ 

Define two maps $S_1$ and $S_2 : \Omega \to \ell^\infty_{n_0}$ as follows.
(S_1 x)(n) = \begin{cases} \frac{3 + c_1 - 3c_2}{4} - \frac{p(n)x(n - \tau)}{4} + \sum_{s=\tau}^{n} q_1(s)x(s - \sigma_1), n \geq n_1, \\ (S_1 x)(n_0), n_0 \leq n \leq n_1. \end{cases}

(S_2 x)(n) = \begin{cases} \sum_{s=\tau}^{n} q_2(s)x(s - \sigma_2) - e(s), n \geq n_1, \\ (S_2 x)(n_0), n_0 \leq n \leq n_1. \end{cases}

(i) We shall show that for any \( x, y \in \Omega \), \( S_1 x + S_2 y \in \Omega \).

For every \( x, y \in \Omega \) and \( n \geq n_1 \), we obtain

\[
(S_1 x)(n) + (S_2 y)(n) \leq \frac{3 + c_1 - 3c_2}{4} - \frac{p(n)x(n - \tau)}{4} + \sum_{s=\tau}^{n} q_1(s)x(s - \sigma_1) + \sum_{s=\tau}^{n} q_2(s)y(s - \sigma_2) - e(s)
\]

\[
\leq \frac{3 + c_1 - 3c_2}{4} + c_2 + \sum_{s=\tau}^{n} |q_1(s)| + M|q_2(s)| + |e(s)|
\]

\[
\leq \frac{3 + c_1 - 3c_2}{4} + c_2 + \frac{3 - c_1 - c_2}{4} = 1.
\]

Furthermore, we have

\[
(S_1 x)(n) + (S_2 y)(n) \geq \frac{3 + c_1 - 3c_2}{4} - \frac{p(n)x(n - \tau)}{4} + \sum_{s=\tau}^{n} q_1(s)x(s - \sigma_1) + \sum_{s=\tau}^{n} q_2(s)y(s - \sigma_2) - e(s)
\]

\[
\geq \frac{3 + c_1 - 3c_2}{4} - c_1 - \sum_{s=\tau}^{n} |q_1(s)| + M|q_2(s)| + |e(s)|
\]

\[
\geq \frac{3 + c_1 - 3c_2}{4} - c_1 - \frac{1 - c_1 - c_2}{4} = \frac{1 - c_1 - c_2}{4}.
\]

Hence

\[
\frac{1 - c_1 - c_2}{2} \leq (S_1 x)(n) + (S_2 y)(n) \leq 1 \quad \text{for} \quad n \geq n_0.
\]

Thus we have proved that \( S_1 x + S_2 y \in \Omega \) for any \( x, y \in \Omega \).

(ii) We shall show that \( S_1 \) is a contraction mapping on \( \Omega \). In fact for \( x, y \in \Omega \) and \( n \geq n_1 \) we have

\[
\|(S_1 x)(n) - (S_1 y)(n)\| \leq |p(n)||x(n - \tau) - y(n - \tau)| + \sum_{s=\tau}^{n} q_1(s)|x(n - \sigma_1) - y(n - \sigma_1)|
\]

\[
\leq \left(p(n) + \sum_{s=\tau}^{n} |q_1(s)| \right) \|x - y\|
\]

\[
\leq \left(c_1 + \frac{1 - c_1}{2}\right) \|x - y\|
\]

\[
\|S_1 x - S_1 y\| \leq \frac{1 + c_1}{2} \|x - y\|.
\]

We conclude that \( S_1 \) is a contraction mapping on \( \Omega \).
(iii) We now show that $S_2$ is completely continuous.

First, we shall show that $S_2$ is continuous. Let $x_k(n)$ be a sequence in $\Omega$ such that $x_k(n) \to x(n)$ as $k \to 1$. Since is closed, $x = \{x(n)\} \in \Omega$.

For $n \geq n_1$, we have,

$$|(S_2x_k)(n) - (S_2x)(n)| \leq \sum_{s = n_1}^{\infty} |q_2(n)||x_k(s - \sigma_2)x_k(s) - x(s - \sigma_2)x(s)|$$

Since

$$|x_k(s - \sigma_2)x_k(s) - x(s - \sigma_2)x(s)| \to 0 \quad \text{as} \quad k \to \infty,$$

by applying the Lebesgue dominated convergence theorem we obtain that

$$\lim_{k \to \infty} ||(S_2x_k)(n) - (S_2x)(n)|| = 0.$$ 

This means that $S_2$ is continuous.

Next, we shall we prove that $(S_2x)(n)$ is uniformly Cauchy. For any given $\epsilon > 0$, there exists $n_1 \geq n_0$ such that

$$\sum_{s = n_1}^{\infty} M|q_2(s)| + |e(s)| \leq \frac{\epsilon}{2}$$

Then for $x = \{x(n)\} \in \Omega$. $n_2 > n_1 \geq n_0$, we get

$$|(S_2x)(n_2) - (S_2x)(n_1)| \leq |(S_2x)(n_2)| + |(S_2x)(n_1)|$$

$$= \sum_{s = n_2}^{\infty} M|q_2(s)| + |e(s)| + \sum_{s = n_1}^{\infty} M|q_2(s)| + |e(s)|$$

$$\leq \frac{\epsilon}{2} + \epsilon = \epsilon.$$

Therefore $(S_2x)(n)$ is uniformly Cauchy. By Lemma 2.3, $(S_2x)(n)$ is relatively compact. By Theorem 2.4, there is $x = \{x_n\} \in \Omega$ such that $S_1x + S_2x = x$. Clearly $x = \{x_n\}$, is a bounded positive solution for equation (1). This completes the proof of the theorem.

**Example 3.2:** Consider the following nonlinear delay difference equation

$$2x(n - 1) + \frac{1}{(n)(n - 2)} x(n - 1) + \frac{1}{(n - 1)(n - 2)} x(n)x(n - 1) = \frac{2n + 3}{2n(n - 1)} - \frac{1}{n + 1}, n \geq 3.$$  

$p(n) = \frac{1}{2}, \sum_{s = n_0}^{\infty} q_i(s) < \infty, i = 1, 2$. Since conditions of Theorem 3.1 are satisfied, the equation (4) has a nonoscillatory bounded positive solution.
Example 3.3: Consider the following nonlinear delay difference equation

\[
\Delta \left( x(n) - \frac{1}{2} x(n-1) \right) + \frac{1}{n^2} x(n-1) + \frac{1}{n(n-1)} x(n) x(n-1) = \frac{1}{n+1} + \frac{3-2n}{2n(n-1)} + \frac{1}{n(n-1)}, n \geq 2. \tag{5}
\]

\[p(n) = -\frac{1}{2}, \quad \sum_{s=\sigma _0}^{\infty } q_i(s) < \infty, \quad i = 1, 2\]

Since conditions of Theorem 3.1 are satisfied, then the equation (5) has a nonoscillatory bounded positive solution.
REFERENCES


