CONSTRUCTION OF FIVE-LEVEL MODIFIED THIRD ORDER ROTATABLE DESIGN USING A PAIR OF BALANCED INCOMPLETE BLOCK DESIGNS.

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Abstract: In this paper, new methods of construction of five level modified third order rotatable designs (TORD) using two suitably chosen balanced incomplete block designs are suggested. It is observed that the method sometimes leads to designs with less number of design points than those available in the literature.

Key words: Rotatable designs, third order rotatable designs, response surfaces, balanced incomplete block designs, five level designs.

1. INTRODUCTION

The study of rotatable designs mainly emphasized on the estimation of absolute response. Box and Hunter (1957) introduced rotatable designs for the exploration of response surfaces considering the variances of the estimated response constant at points equidistant from the centre of the design. Response surface methodology is a collection of mathematical and statistical techniques useful for developing, improving and optimizing models and processes in which a response of interest might be influenced by several variables. In many experimental situations the experimenter is concerned with explaining certain aspects of a functional relationship, \(Y = f(x_1, x_2, \ldots, x_p) + e\), where \(Y\) is the response, \(x_1, x_2, \ldots, x_p\) are \(p\) factors and \(e\) is the random error. The function \(f(.)\) is called response surface or response function. Designs, which are used, for the study of response surface methods, are called response surface designs. Response surface methods are useful where several independent variables influence a dependent variable. The independent variables are assumed to be continuous and controlled by the experimenter. The response is assumed to be a random variable. The independent variables are often called input or explanatory variables and the dependent variable is often called the response variable. The fitting of the response surface can be complex and costly if done haphazardly. To cut on costs, an experimenter has to make a choice of the experimental design prior to experimentation.

Box and Hunter (1957) introduced rotatable designs for the exploration of response surfaces. Many third order rotatable designs have been described in Gardiner et al. (1959), Draper (1960a, b, 1961), Thaker and Das (1961), Herzberg (1964), Tyagi (1964), and Nigam (1967). These designs would usually require many more points than the available minimal point designs and hence may not always be desirable. Das and Narasimham (1962) constructed rotatable designs through balanced incomplete block designs (BIBD). Narasimham et al. (1983) constructed second order rotatable designs (SORD) using a pair of incomplete block designs. For some new sequential methods, the reader is referred to Huda (1982b, 1983) and Mutiso and Koske (2005, 2006). Victorbabu and Narasimham (1995a) suggested three levels SORD using a pair of BIBDS. Das et al. (1999) studied response surface designs, symmetrical and asymmetrical rotatable and modified. Koske et al (2011) introduced a new method of constructing higher level of third order rotatable designs using BIBDS. Victorbabu and Vasundharadevi (2005b) studied modified second order response surface designs using BIBD. Victorbabu (2006) suggested new methods of construction of three and five-level modified second order rotatable designs (SORD) and modified second order slope rotatable designs (SOSRD) using suitably chosen balanced incomplete block designs. Victorbabu (2009) examined in detail different methods of construction of modified second order response surface designs, modified second order rotatable designs (SORD), modified SORD with equispaced doses (levels) using central composite designs, balanced incomplete block designs (BIBD), pairwise balanced designs (PBD) and symmetrical unequal block arrangements (SUBA) with two unequal block sizes. Victorbabu (2011) explored a new method of construction of second-order slope-rotatable designs using incomplete block designs with unequal block sizes. Further, Victorbabu and Rajyalakshmi (2012) studied a new method of construction of robust second order rotatable designs using balanced incomplete block designs. In this study, we examined the methods of construction of three-level modified third order rotatable designs (TORD) using balanced incomplete block designs (BIBD).
2. REVIEW OF MOMENT CONDITIONS FOR THIRD ORDER ARRANGEMENT TO BE ROTATABLE.

Suppose we want to use the third order response surface design \( D = \{x_{iu}\} \) to fit the surface,
\[
Y_u = \beta_0 x_{0u} + \sum_{i=1}^{k} \beta_i x_{iu} + \sum_{i<j}^{k} \beta_{ij} x_{iu} x_{ju} + \sum_{i<j<k}^{k} \beta_{ijk} x_{iu} x_{ju} x_{ku} + e_u
\]
(1)

Where \( x_{iu} \) denotes the level of the \( i \)th factor \((i = 1, 2, \ldots, k)\) in the \( u \)th run \((u = 1, 2, \ldots, N)\) of the experiment, \( e_u \)'s are uncorrelated random errors with mean zero and variance sigma squared. Here \( \beta_0, \beta_i, \beta_{ij}, \beta_{ijk}, \) and \( \beta_{ij} \) are the parameters of the model and \( Y_u \) is the response observed at the \( u \)th design point. The parameters in the response relation are estimated using the least squares technique.

Further, we impose the following simple symmetry conditions on the design points to simplify the solutions of the normal equations.
\[
\sum_{u=1}^{N} N_i x_{iu}^2 = 0 \quad \text{if any} \quad u_i \text{ is odd for } \sum_{i=1}^{k} u_i = 6.
\]
(2)

\[
\begin{align*}
(i) & \quad \sum_{u=1}^{N} x_{iu}^2 = N \lambda_2, \quad \forall i \\
(ii) & \quad \sum_{u=1}^{N} x_{iu}^4 = dN \lambda_4, \quad \forall i \\
(iii) & \quad \sum_{u=1}^{N} x_{iu}^2 x_{ju} = N \lambda_4, \quad i \neq j \\
(iv) & \quad \sum_{u=1}^{N} x_{iu}^6 = hN \lambda_6, \quad \forall i \\
(v) & \quad \sum_{u=1}^{N} x_{iu}^2 x_{ju}^2 = cN \lambda_6, \quad i \neq j \neq 1 \\
(vi) & \quad \sum_{u=1}^{N} x_{iu}^2 x_{ju} x_{ku} = N \lambda_6, \quad 1 \neq j \neq k
\end{align*}
\]

where \( c, d, h, \lambda_2, \lambda_4, \text{ and } \lambda_6 \) are constants, \( x_{0u} = 1, \ E(e_u) = 0, \text{ var}(e_u) = \sigma^2 \) (unknown), \( \text{cov}(e_u, e_{u'}) = 0, \ u \neq u' = 1, 2, \ldots, N. \)

Using these symmetry conditions, we obtain the estimates of the parameters as,
\[
\beta_0 = \frac{(k + 2) \lambda_4 \sum y_u - k \lambda_2 \sum x_{iu}^2 y_u}{N[(k + 2) \lambda_4 - k \lambda_4^2]}
\]
\[
\beta_i = \frac{[(k + 1) \lambda_4 - (k - 1) \lambda_4^2] \sum x_{iu}^3 y_u + (k - 1) (\lambda_4^2 - \lambda_4) \sum_{i=2}^{k} x_{iu}^3 y_u - 2 \lambda_4 \lambda_2 \sum y_u}{2N \lambda_4 [(k + 2) \lambda_4 - k \lambda_4^2]} \]
\[
\beta_{ij} = \frac{\sum x_{iu} x_{ju} y_u}{N \lambda_4}
\]
\[
\beta_{ij} = \frac{(c + k - 1) \lambda_6 \sum x_{iu} y_u - \lambda_4 \sum x_{iu}^3 y_u - (k - 1) \lambda_4 \sum \left[ \sum x_{iu} x_{ju}^2 y_u \right]}{N[(c + k - 1) \lambda_6 \lambda_2 - (d + k - 1) \lambda_4^2]}
\]
\[
\beta_{iij} = \frac{[(k + 1) \lambda_6 \lambda_2 - (k - 1) \lambda_4^2] \sum x_{iu}^3 y_u - (c - 2)(k - 1)(\lambda_4^2 - \lambda_6 \lambda_2) \sum \sum \left[ \sum x_{iu} x_{ju}^2 y_u \right] - (c + 1) \lambda_6 \lambda_4 \sum x_{iu} y_u}{(c + 1) N \lambda_6 [(c + k - 1) \lambda_6 \lambda_2 - (d + k - 1) \lambda_4^2]} \]
\[
\beta_{iiij} = \frac{11}{(c + 1) N \lambda_6 [(c + k - 1) \lambda_6 \lambda_2 - (d + k - 1) \lambda_4^2]}
\]
\begin{equation*}
\left( \lambda_4^2 - \lambda_6 \lambda_2 \right) \sum x_{iu}^2 y_u + [(k + 3) \lambda_6 \lambda_2 - (k + 1) \lambda_4^2] \sum \sum \left( \sum x_{iu} x_{ju}^2 y_u \right) + \\
(k - 2)(\lambda_4^2 - \lambda_6 \lambda_2) \sum \sum \left( \sum x_{iu} x_{ju}^2 y_u \right) - (d - 1) \lambda_6 \lambda_4 \sum x_{iu} y_u
\end{equation*}

\begin{equation*}
\beta_{ij} = \frac{(d - 1) N \lambda_6 \left[ (c + k - 1) \lambda_6 \lambda_2 - (d + k - 1) \lambda_4^2 \right]}{N} \sum x_{iu} x_{ju} x_{iu} y_u
\end{equation*}

and,

\begin{equation*}
\beta_{ij} = \frac{\sum x_{iu} x_{ju} x_{iu} y_u}{N \lambda_6}
\end{equation*}

Using these solutions, the variances and covariance of the estimated response parameters are given below,

\begin{align*}
\text{Var}(\beta_0) &= \frac{(k + 1) \lambda_4}{N[(d + k - 1) \lambda_4 - k \lambda_2^2]} \sigma^2 \\
\text{Var}(\beta_u) &= \frac{(k + 1) \lambda_4 - (k - 1) \lambda_2^2}{2N \lambda_4 [(d + k - 1) \lambda_4 - k \lambda_2^2]} \sigma^2 \\
\text{Var}(\beta_i) &= \frac{1}{N \lambda_4 \sigma^2} \\
\text{Var}(\beta_{ii}) &= \frac{(c + k - 1) \lambda_6}{N[(c + k - 1) \lambda_6 \lambda_2 - (d + k - 1) \lambda_4^2]} \sigma^2, \\
\text{Var}(\beta_{ii}) &= \frac{(k + 1) \lambda_6 \lambda_2 - (k - 1) \lambda_4^2}{(c + 1) N \lambda_6 [(c + k - 1) \lambda_6 \lambda_2 - (d + k - 1) \lambda_4^2]} \sigma^2 \\
\text{Var}(\beta_{ij}) &= \frac{[(c + k - 2) \lambda_6 \lambda_2 - (k + 1) \lambda_4^2]}{(d - 1) N \lambda_6 [(c + k - 1) \lambda_6 \lambda_2 - (d + k - 1) \lambda_4^2]} \sigma^2 \\
\text{Var}(\beta_{jj}) &= \frac{1}{N \lambda_6 \sigma^2} \\
\text{Cov}(\beta_0, \beta_u) &= -\frac{\lambda_2}{N[(d + k - 1) \lambda_4 - k \lambda_2^2]} \sigma^2 \\
\text{Cov}(\beta_{ii}, \beta_{ij}) &= \frac{-(\lambda_4 - \lambda_2^2)}{2N \lambda_4 [(d + k - 1) \lambda_4 - k \lambda_2^2]} \sigma^2 \\
\text{Cov}(\beta_i, \beta_{ii}) &= -\frac{\lambda_4}{N[(c + k - 1) \lambda_6 \lambda_2 - (d + k - 1) \lambda_4^2]} \sigma^2 \\
\text{Cov}(\beta_{ii}, \beta_{jj}) &= \frac{-(\lambda_2^2 - \lambda_6 \lambda_2)}{(d - 1) N \lambda_6 [(c + k - 1) \lambda_6 \lambda_2 - (d + k - 1) \lambda_4^2]} \sigma^2,
\end{align*}

and other covariances are zero.
An inspection of the variances shows that necessary conditions for the existence of a non singular third order design are:

\[(d + k - 1)\lambda_4 - k\lambda_2^2 > 0\] and \[(c + k - 1)\lambda_6 - (k + 2)\lambda_4^2 > 0\], which leads to the conditions,

\[(i) \quad \frac{\lambda_4}{\lambda_2^2} > \frac{k}{d + k - 1}\]

\[(ii) \quad \frac{\lambda_6\lambda_2}{\lambda_4^2} > \frac{k + 2}{c + k - 1}\]  \[\text{[i(i) and (ii) being non-singularity conditions]}\]

3. DEFINITION OF THIRD ORDER ROTATABLE DESIGN (TORD)

A third order response surface design D is said to be a third order rotatable design, if in this design, \(d=3\), \(c=5\) and all the other conditions (2) to (11) hold and hence the variance of the estimated response \(Y_u\) from the fitted surface is only a function of the distance of the point \((x_1, x_2, \ldots, x_k)\) from the center of the design, Box and Hunter (1957).

4. MODIFIED THIRD ORDER ROTATABLE DESIGN

The usual method of constructing a third order rotatable design is by putting some restrictions indicating some relation among the third order moments,

\[\sum x_{iu}^2, \sum x_{iu}^4, \sum x_{iu}^2 x_{ju}^2, \sum x_{iu}^6, \sum x_{iu}^4 x_{ju}^2, \text{ and } \sum x_{iu}^2 x_{ju}^2 x_{ku}^2.\]

Some equations involving the unknowns are obtained and their solutions give the unknown levels. In third order rotatable designs, the restrictions used are; \(\sum x_{iu}^4 = 3 \sum x_{iu}^2 x_{ju}^2\) and \(\sum x_{iu}^6 = 5 \sum x_{iu}^4 x_{ju}^2 = 15 \sum x_{iu}^2 x_{ju}^2 x_{ku}^2\). Other restrictions which may have not been exploited could also be possible. In the present study, we investigated the following restrictions:

\((\sum x_{iu}^4)^2 = N \sum x_{iu}^2 x_{ju}^2\), \(i.e., \lambda_4^2 = \lambda_4\) and \((\sum x_{iu}^2 x_{ju}^2)^2 = N (\sum x_{iu}^2 x_{ju}^2 x_{ku}^2)^2\), \(i.e., \lambda_4^2 \lambda_6 = \lambda_4^2\). These gives another series of spherical third order response surface designs which provides more precise estimates of the response at specific points of interest than what is available from the corresponding existing designs. Further, the variances and covariances of the estimated parameters are,

\[\text{Var}(\hat{\beta}_0) = \frac{(k + 1)\sigma^2}{2N}\]

\[\text{Var}(\hat{\beta}_{ij}) = \frac{1}{(d - 1)N\lambda_4}\sigma^2\]

\[\text{Var}(\hat{\beta}_i) = \frac{1}{N\lambda_4}\sigma^2\]

\[\text{Var}(\hat{\beta}_i) = \frac{(c + k - 1)\lambda_6}{2N\lambda_4^2}\sigma^2,\]

\[\text{Var}(\hat{\beta}_{ii}) = \frac{1}{(c + 1)N}\sigma^2\]

\[\text{Var}(\hat{\beta}_{ij}) = \frac{1}{2N}\sigma^2\]

\[\text{Var}(\hat{\beta}_{ij}) = \frac{1}{N\lambda_6}\sigma^2\]
\[ \text{Cov}(\beta_0, \beta_i) = -\frac{1}{2N\sqrt{\lambda_i}} \sigma^2 \]
\[ \text{Cov}(\beta_i, \beta_{i+}) = -\frac{1}{2N\lambda_i} \sigma^2 \]

and all other co-variances are zero.

5. FIVE-LEVEL MODIFIED THIRD ORDER ROTATABLE DESIGNS USING A PAIR OF BIBD.

The method of construction of five level modified TORD using a suitably chosen pair of balanced Incomplete block designs (BIBDs) without any additional set of points was obtained.

**Definition**

Let \( D_i = (v, b_i, r_i, k_i, \lambda_i) \), for \( i = 1, 2 \) be two BIBDs. \( 2^{e(k_i)} \) denotes resolution V fractional replicates of \( 2^{k_i} \) factorials with +1 or -1 levels in treatments with \( r_i \leq 5\lambda_i \) and \( r_i \geq 5\lambda_i \) respectively.

Let \( [1-(v, b_i, r_i, k_i, \lambda_i)]2^{e(k_i)} \) and \( [a-(v, b_i, r_i, k_i, \lambda_i)]2^{e(k_i)} \) denote the \( b_12^{e(k_i)} \) and \( b_22^{e(k_i)} \) design points generated from the BIB designs by multiplication respectively. The set of \( b_12^{e(k_i)} \) design points generated from BIBD-D_1 is repeated \( m_1 \) times and the set of \( b_22^{e(k_i)} \) design points generated from the BIBD-D_2 is repeated \( m_2 \) times.

Let \( n_0 \) denote the number of central points. Then with the above design points, \( m_1 b_12^{e(k_i)} \), we construct a five level modified TORD as given in the following theorem.

**Theorem 1**

The design points

\[ m_1[1-(v, b_i, r_i, k_i, \lambda_i)]2^{e(k_i)} \cup m_2[a-(v, b_i, r_i, k_i, \lambda_i)]2^{e(k_i)} \cup n_0 \]

give a five level \( v \)-dimensional modified TORD in,

\[ N = \frac{[m_1r_12^{e(k_i)} + m_2r_22^{e(k_i)}]a^2}{[m_1\lambda_12^{e(k_i)} + m_2\lambda_22^{e(k_i)}]a} \] .............................. (12)

design points if;

\[ (r_1 - 5\lambda_1)(r_2 - 5\lambda_2) \leq 0. \]

\( a^6 = \frac{m_1(r_1-5\lambda_1)2^{e(k_i)-o(k_i)}}{m_2(5\lambda_2-r_2)} \) .............................. (13)

\( a_6 = \frac{(m_1r_12^{e(k_i)} + m_2r_22^{e(k_i)})a^2}{[m_1\lambda_12^{e(k_i)} + m_2\lambda_22^{e(k_i)}]a^2} - m_1b_12^{e(k_i)} - m_2b_22^{e(k_i)} \) .............................. (14)

and \( n_0 \) turns out to be an integer.

**Proof**

For the design points generated, \( m_1 \) repetitions of points from BIBD-D_1 and \( m_2 \) repetitions of points from BIBD-D_2, the conditions for the modified TORD are true as follows:

\[ \sum x^{2}_{il} = m_1r_12^{e(k_i)} + m_2r_22^{e(k_i)}a^2 = A \] .............................. (15)

\[ \sum x^{4}_{il} = m_1r_12^{e(k_i)} + m_2r_22^{e(k_i)}a^4 = 3B \] .............................. (16)
\[ \sum x_{iu}^{2} = m_{1} \lambda_{1} 2^{t(k_{2})} + m_{2} \lambda_{2} 2^{t(k_{2})} \alpha^{4} = B \] \[ \sum x_{iu}^{6} = m_{1} \lambda_{1} 2^{t(k_{2})} + m_{2} \lambda_{2} 2^{t(k_{2})} \alpha^{6} = 15C \] \[ \sum x_{iu}^{4} = m_{1} \lambda_{1} 2^{t(k_{2})} + m_{2} \lambda_{2} 2^{t(k_{2})} \alpha^{6} = 3C \]

where \( N \lambda_{2} \cdot B = N \lambda_{2} \) and \( C = N \lambda_{6} \).

From (18) and (19), we have,
\[ m_{1} \lambda_{1} 2^{t(k_{2})} + m_{2} \lambda_{2} 2^{t(k_{2})} \alpha^{6} = 5 \left[ m_{1} \lambda_{1} 2^{t(k_{2})} + m_{2} \lambda_{2} 2^{t(k_{2})} \alpha^{6} \right] \]

which leads to,
\[ a^{6} = \frac{m_{1} \lambda_{1} 2^{t(k_{2})} + m_{2} \lambda_{2} 2^{t(k_{2})} \alpha^{6}}{m_{2} (5 \lambda_{2} - \lambda_{1})} \]

given in equation (13).

The modified condition \( \sum x_{iu}^{2} = N \sum x_{iu}^{2} \) leads to
\[ N = \frac{\left[ m_{1} \lambda_{1} 2^{t(k_{2})} + m_{2} \lambda_{2} 2^{t(k_{2})} \alpha^{6} \right]^{2}}{\left[ m_{1} \lambda_{1} 2^{t(k_{2})} + m_{2} \lambda_{2} 2^{t(k_{2})} \alpha^{6} \right]^{2}} \]

given in equation (12).

Given \( n_{0} \) central points, \( N \) may be obtained directly as
\[ N = m_{1} b_{1} 2^{t(k_{1})} + m_{2} b_{2} 2^{t(k_{2})} + n_{0} \]

Example

The design points
\[ m_{1} [1 - (v = 7, b_{1} = 7, r_{1} = 3, k_{1} = 3, \lambda_{1} = 1)] 2^{3} \cup \]
\[ m_{2} [\alpha - (v = 7, b_{2} = 21, r_{2} = 6, k_{2} = 2, \lambda_{2} = 1)] 2^{3} \cup n_{0} \]
give a five level 7-dimensional modified third order rotatable design in
\[ N = 288 \]
design points with \( m_{1} = 2, m_{2} = 1 \). Here (13) leads to \( a = \sqrt{2} \) and (14) leads to \( n_{0} = 92 \).

6. FIVE-LEVEL MODIFIED THIRD ORDER ROTATABLE DESIGNS USING A PAIR OF DBIBD.

The method of construction of three level modified TORD using a suitably chosen pair of doubly balanced Incomplete block designs (DBIBDs) without any additional set of points was obtained.

Definition

Let \( D'_{i} = (v_{i}, b_{i}, r_{i}, k_{i}, \lambda_{i}, \mu_{i}) \) for \( i = 1, 2 \) be two doubly balanced incomplete block designs (DBIBDs), \( 2^{t(k_{i})} \) denotes resolution V fractional replicates of \( 2^{k_{i}} \) factorials with +1 or -1 levels in treatments with either \( \lambda_{i} \leq 3 \mu_{i} \) and \( \lambda_{i} \geq 3 \mu_{i} \) or \( r_{1} \leq 15 \mu_{2} \) and \( r_{2} \geq 15 \mu_{2} \) respectively.

Let \( [1 - (v_{i}, b_{i}, r_{i}, k_{i}, \lambda_{i}, \mu_{i})] 2^{t(k_{i})} \) and \( [\alpha - (v_{i}, b_{i}, r_{i}, k_{i}, \lambda_{i}, \mu_{i})] 2^{t(k_{i})} \) denote the \( b_{i} 2^{t(k_{i})} \) and \( b_{i} 2^{t(k_{i})} \) design points generated from the DBIBD- \( D'_{1} \) and DBIBD- \( D'_{2} \) designs by multiplication respectively. The set of \( b_{i} 2^{t(k_{i})} \) design points generated from DBIBD- \( D'_{1} \) is repeated \( m'_{1} \) times and the set of \( b_{i} 2^{t(k_{i})} \) design points generated from the DBIBD- \( D'_{2} \) is repeated \( m'_{2} \) times.

Let \( n'_{0} \) denote the number of central points. Then with the above design points \( m'_{1} b_{i} 2^{t(k_{i})} \), we construct a five level modified TORD as given in the following theorem.
Theorem
The design points $m'_1[1 - (v, b_1, r_1, k_1, \lambda_1, \mu_1)]2^{tk_2} \cup m'_2[v - (v, b_2, r_2, k_2, \lambda_2, \mu_2)]2^{tk_2} \cup n'_0$ give a five level v-dimensional modified TORD in,
\[ N = \frac{[m'_1 \lambda_2 2^{tk_2} + m'_2 \lambda_2 2^{tk_2}]^2}{[m'_1 \lambda_2 2^{tk_2} + m'_2 \lambda_2 2^{tk_2}]2^{tk_2}} \]  

(20)

design points if either,
\[(\lambda_1 - 3\mu_1)\lambda_2 - 3\mu_2, \leq 0 \]
\[a^6 = \frac{m'_1(\lambda_1 - 3\mu_1)2^{tk_2} - \lambda_2}{m'_2(3\mu_2 - \lambda_2)} \]  

(21)
or \((r_1 - 15\mu_1)(r_2 - 15\mu_2) \leq 0 \)
\[a^6 = \frac{m'_1(r_1 - 15\mu_1)2^{tk_2} - \lambda_2}{m'_2(15\mu_2 - r_2)} \]  

(22)

\[n'_0 = \frac{[m'_1 \mu_2 2^{tk_2} + m'_2 \mu_2 2^{tk_2}]2^{tk_2}}{[m'_1 \mu_2 2^{tk_2} + m'_2 \mu_2 2^{tk_2}]2^{tk_2}} - m'_1 b_1 2^{tk_2} - m'_2 b_2 2^{tk_2} \]  

(23)

and $n'_0$ turns out to be an integer.

Proof
For the design points generated, $m'_1$ - repetitions of points from DBIBD-$D'_1$ and $m'_2$ - repetitions of points from DBIBD-$D'_2$, the conditions for the modified TORD are true as follows:

\[ \sum x_{iu}^2 = m'_1 r_1 2^{tk_2} + m'_2 r_2 2^{tk_2} \alpha^2 = A' \]  

(5.4.5)  

(24)

\[ \sum x_{iu}^4 = m'_1 r_1 2^{tk_2} + m'_2 r_2 2^{tk_2} \alpha^4 = 3B' \]  

(25)

\[ \sum x_{iu}^6 = m'_1 r_1 2^{tk_2} + m'_2 r_2 2^{tk_2} \alpha^6 = B' \]  

(26)

\[ \sum x_{iu}^8 = m'_1 r_1 2^{tk_2} + m'_2 r_2 2^{tk_2} \alpha^8 = 15C' \]  

(27)

\[ \sum x_{iu}^4 x_{ju}^2 = m'_1 r_1 2^{tk_2} + m'_2 r_2 2^{tk_2} \alpha^6 = 3 \]  

(28)

\[ \sum x_{iu}^4 x_{ju}^4 = m'_1 r_1 2^{tk_2} + m'_2 r_2 2^{tk_2} \alpha^8 = 3 \]  

(29)

where $A' = N \lambda_2$, $B' = N \lambda_4$, and $C = N \lambda_6$.

From (28) and (29), we have,
\[ [m'_1 r_1 2^{tk_2} + m'_2 r_2 2^{tk_2} \alpha^6] = 3 [m'_1 r_1 2^{tk_2} + m'_2 r_2 2^{tk_2} \alpha^8] \]

which leads to,
\[ a_0 = \frac{m_1(\lambda_1-3\mu_1)\alpha(\lambda_1-\mu_1)}{m_1(3\lambda_1-\mu_1)} \], given in equation (21).

Further, from (27) and (29) we have,

\[ m'_1 r_12^{(2^k)} + m'_2 r_22^{(2^k)} \alpha^6 = 15[m'_1 \mu_1 2^{(2^k)} + m'_2 \mu_2 2^{(2^k)} \alpha^6] \]

which leads to,

\[ a_0 = \frac{m'_1 (r_1-15\mu_1)2^{(2^k)}-4(\mu_1)}{m'_2 (15\mu_2-r_2)} \] given in (22).

The modified condition \((\sum x_{2i}^2 x_{2i}^2)^3 = N (\sum x_{2i}^2 x_{2i}^2 x_{2i}^2)^2\) leads to

\[ N = \frac{[m'_1 \lambda_1 2^{(2^k)}+m'_2 \lambda_2 2^{(2^k)} \alpha^6]^{3}}{[m'_1 \mu_1 2^{(2^k)}+m'_2 \mu_2 2^{(2^k)} \alpha^6]^{2}} \], given in equation (20).

Given \(n'_0\) central points, \( N \) may be obtained directly as

\[ N = m'_1 b_1 2^{(2^k)} - m'_2 b_2 2^{(2^k)} + n'_0. \]

Example
The design points

\[ m'_1 [1 -(v = 5, b_1 = 5, r_1 = 4, k_1 = 4, \lambda_1 = 3, \mu_1 = 2 )]2^4 U \]

\[ m'_2 [\alpha -(v = 5, b_2 = 10, r_2 = 4, k_2 = 2, \lambda_2 = 1, \mu_2 = 0)]2^4 U n'_0 \]

give a five level 5-dimensional modified third order rotatable design in

\( N = 256 \) design points with \( m'_1 = 1, m'_2 = 1 \). Here (22) leads to \( \alpha = \sqrt{2} \) and (23) leads to \( n_0 = 136 \).

7. CONCLUSION
During the last few decades, rotatable designs using balanced incomplete block designs (BIBD) have been discussed by several authors. This introduces a new method of constructing a five level modified third order rotatable designs using (BIBD) and doubly balanced incomplete block designs (DBIBD).

REFERENCES


