NUMERICAL SOLUTION OF LOGISTIC DIFFUSION EQUATION AND COMPETING SPECIES ANALYSIS OF TUMOR GROWTH

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The present Chapter deals with two models for the analysis of tumor growth one is competing species system and other logistic diffusion equation system. A simple model of tumor describes the growth of two populations, each growing according to a logistic law and competing with each other. In this model we lump together all non-tumor cells which are at tumor site, including normal tissue as well as immune cells. Here immune cells source do not assume constant. Another model which is based on logistic diffusion system is considered. We have studied the numerical solutions for tumor cell density for constant and time-dependent growth rate. Graphical representation for the solution has been also illustrated.

KEY WORDS: Logistic-diffusion equation, Tumor cell density, Growth rate, Carrying capacity.

1. INTRODUCTION

Tumor growth is a complex process. It has been a challenge for many years to search a suitable growth law to uncover biological complexities. Mathematical models based of simple mathematical equations, such as deterministic exponential, logistic and Gompertzian equations are used as a basic tools for describing tumor growth.


Kubo (2010) discussed solvability and asymptotic profile of the solution to some parabolic ode system described tumor angiogenesis. Meral (2010) considered the density dependent nonlinear reaction diffusion equation which arises in the insect dispersal models and has been solved using combined application of differential quadrature method (DTM) and implicit Euler method. Wang et al., (2011) proposed a partial differential equation (PDE), especially diffusive logistic (DL) equation to model the temporal and spatial characteristics of information diffusion in online social networks.

Model 1: Competing Species System

Let $x(t)$ and $y(t)$ be the normal cell population (including immune cells) and tumor cell population respectively at time $t$. The system of differential equations which describe the model is

$$\frac{dx}{dt} = p_1 x(1 - q_1 x) - r_1 xy$$

$$\frac{dy}{dt} = p_2 y(1 - q_2 y) - r_2 xy$$

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\( p_1, p_2 \): Growth rates of normal and tumor cells
\( q_1, q_2 \): Carrying capacity of normal and tumor cells
\( r_1, r_2 \): Competition rate parameters of the normal cells and tumor cells

\( p_1, p_2, q_1, q_2, r_1, r_2 > 0 \), Since the specified system of equations has the required negatives signs to account for decreases in numbers whenever necessary.

For the case that tumor-free equilibrium \((1/q_1, 0)\) is to be stable, liberalize about this equilibrium point \((x = 1/q_1 + \delta \lambda, y = 0 - \delta \mu)\) where \(\delta\) is small compared to \(1/q_1\).

\[
\begin{bmatrix}
\lambda' \\
\mu'
\end{bmatrix} =
\begin{bmatrix}
-p_1 & \frac{r_1}{q_1} \\
0 & p_2 \left(1 - \frac{r_2}{q_1 p_2}\right)
\end{bmatrix}
\begin{bmatrix}
\lambda \\
\mu
\end{bmatrix}.
\]

(3)

It is seen from eqn. (3) tumor can not recur if the Eigen value \(1 - \frac{r_2}{q_1 p_2}\) is negative and thus the condition for the tumor free equilibrium to be stable is \(\frac{r_2}{q_1 p_2} > 1\).

Figure1: Four Cases for the Competiting Species System
Model 2: Logistic diffusion Equation:

Let \( u(x, t) \) be the density of tumor cell density at any time \( t \) and \( r \) be the growth rate of tumor cells and \( k \) be the carrying capacity. Then \( u(x, t) \) satisfies the logistic diffusion equation

\[
\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} + r(t)u \left( 1 - \frac{u(x)}{k} \right).
\]  

(4)

Subject to the initial condition \( u(x, 0) = e^{0.1x} \) and solution satisfies the boundary conditions \( u(0, t) = 0 \) and \( e^{-t} + \int_{0}^{x} \frac{\partial u}{\partial x} dx, x(1, t) = 0 \).

The parameter \( D > 0 \) corresponds to the rate at which tumor cell density diffuses.

The term \(-u^2\) in the equation corresponds to the fact that the tumor cell population is self-limiting and function \( r \) correspond to birth rate of tumor cell population if self limitation is ignored.

2. Numerical Solution

Figure 2: Numerical Solution Computed with 50 Mesh Points for \( r = \sin t \)

Figure 3: Numerical Solution Computed with 50 Mesh Points for \( r = t \)
Figure 4: Numerical Solution Computed with 50 Mesh Points for $r = 0.6$

Figure 5: Solution at $t = 2$ for $r = \sin t$

Figure 6: Solution at $t = 2$ for $r = t^2$
3. Conclusions

In this Chapter, an attempt has to be made to solve logistic diffusion equation for tumor cell population. We have developed a matlab programme for solving this equation numerically. Variations in tumor cell density for constant and time dependent growth rate of tumor cells has been shown in Figures 2, 3, 4. Another model which follows competing species system between normal cells and tumor cells has also been considered. Condition for tumor free equilibrium has been found in this case.

Appendix

Matlab Programme for Solving Logistic Diffusion Equation

function [c, f, s] =eqn(x, t, u, DuDx, r,k)
  r=sint;
  k=150*10^7;
  c = 1;
  f = DuDx;
  s = r*u-r*u^2/k;

function value =initial1(x)
  Value=exp (0.1*x);

function [pl,ql,pr,qr] =bc1(xl,ul,xr,ur,t)
  pl = ul;
  ql = 0;
  pr =exp (-t);
  qr = 1;

m = 0;
  x = linspace (0, 10,200);
  t = linspace (0, 20,100);
  u = pdepe (m, @eqn, @initial1, @bc1, x, t);
  Surf(x, t, u)
  title ('Numerical solution computed with 50 mesh points')
  xlabel ('Distance x')
  ylabel ('Time t')
  plot(x, u (end, :))
  title ('Solution at t = 2')
  xlabel('Distance x')
  ylabel ('u(x, 2)')

References


